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S-Divisibility Property and a Holmstedt Type Formula

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Given a cone P of positive functions and an operator $S: \mathscr{U} \to P$ with $\mathscr{U}$ an additive group, we extend the concept of K-divisibility to get some new formulas for the K-functional of finite families of lattices. Applications are given in the setting of rearrangement invariant spaces and weighted Lorentz spaces. As a consequence of our results, we also obtain a generalized Holmstedt formula due to Asekritova (1980, *Yaroslav. Gos. Univ.* **165**, 15–18). © 1999 Academic Press

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1. INTRODUCTION

Let $\bar{A} := (A_0, A_1)$ be a compatible couple of quasi-normed spaces and let us consider the K-functional (see [7]),

$$K(t, a; \bar{A}) = \inf\{ \|a_0\|_{A_0} + t \|a_1\|_{A_1}: a = a_0 + a_1\}.$$

For $0 < \theta < 1$ and $0 < q \leqslant \infty$ the classical real interpolation spaces are constructed as

$$(A_0, A_1)_{\theta, q} = \{a \in A_0 + A_1: K(\cdot, a; \bar{A}) \in L_q^{\theta}\},$$

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where L_q^θ is the weighted L_q space with weight $t^{-\theta}$ with respect to the measure dt/t and

$$\|a\|_{\theta, q} := \|K(\cdot, a; \bar{A})\|_{L_q^\theta} = \left(\int_0^\infty [t^{-\theta} K(t, a; \bar{A})]^q \frac{dt}{t} \right)^{1/q}.$$

One of the most important formulas that connects these spaces and reiteration theorems is the so-called Holmstedt formula, see [7]. This formula relates the K-functional of the couple $((A_0, A_1)_{\theta_0, q_0}, (A_0, A_1)_{\theta_1, q_1})$ with the K-functional of the function $K(\cdot, a; \bar{A})$ itself with respect to the couple $(L_{q_0}^{\theta_0}, L_{q_1}^{\theta_1})$.

There have been many important extensions of Holmstedt's formula. Let us mention, for example, those concerning K-spaces: given L_0 the space of real valued Lebesgue measurable functions on $\mathbb{R}^+ = (0, \infty)$ and given a lattice $E \subset L_0$ so that $\min(1, t) \in E$, the K-space, defined by

$$(A_0, A_1)_{E; K} = \{a \in A_0 + A_1 : K(\cdot, a; \bar{A}) \in E\},$$

is a quasi-normed space under the norm $\|a\|_{\bar{A}_{E; K}} := \|K(\cdot, a; \bar{A})\|_E$. Then, under some conditions on the couple $\bar{E} := (E_0, E_1)$, the following formula is obtained in [25]:

$$K(t, a; \bar{A}_{E_0; K}, \bar{A}_{E_1; K}) \approx K(t, K(\cdot, a; \bar{A}); E_0, E_1). \tag{1}$$

In order to prove such a formula, one needs to use the fact that every couple $\bar{A}$ is K-divisible, see [8].

The concavity property of the K-functional is fundamental for this theory. Now, if we change the cone of concave functions by the cone of decreasing functions in $\mathbb{R}^+$, we may deal with other classes of spaces including rearrangement invariant spaces. For example, given a function space E, we can define E^* to be the space of all measurable functions so that $f^* \in E$ with $\|f\|_{E^*} = \|f^*\|_E$, and hence one may try to find the connection between the K-functional $K(t, f; E_0^*, E_1^*)$ and $K(t, f^*; E_0, E_1)$. Many results have already been obtained in this direction (see [1, 19, 20, 22–25, 29, 30]), and in order to prove them one has to deal with some kind of divisibility property; see [25].

The situation is much different if we are interested in a finite family $\bar{A} = (A_0, ..., A_n)$ instead of a couple, since there are very few examples of finite families satisfying the K-divisibility property, see [8]. It was shown by Asekritova that even the simple family of weighted L_1 spaces is not K-divisible, see [8, p. 676]. Moreover, Asekritova and Krugljak [4] have recently proved the somewhat surprising result that in the case of finite families of functional Banach lattices the analogue of the equivalence theorem

$$K_{\theta,q} = J_{\theta,q}$$

in fact holds. The proof of this result is based on an analogue of the classical fundamental lemma (with Calderón operator). This result contains implicitly the interesting fact that $(n+1)$-tuples of functional Banach lattices possess the K-divisibility with the squared Calderón operator. This fact shows that it can be of interest to study K-divisibility with "operators," cf. the discussion on weak K-divisibility in Section 2.

The above considerations lead us to extend the concept of K-divisibility to the setting of an operator

$$S: \mathscr{U} \to P,$$

where $\mathscr{U}$ is an additive group and P is a cone of positive functions, which works not only for couples but also for finite families. Although for our applications it will be enough to consider the cone of concave and the cone of decreasing functions, we think that this setting makes things easier and, on the other hand, it has other applications of independent interest (see [9]). Moreover, we note that Asekritova has in [2] stated (without proof) formulas, like those in our Theorems 5 and 6. We obtain these just as a consequence of our results from Section 2.

Let us establish some notation. As usual, $L_0(\Omega)$ denotes the set of all real-valued measurable functions defined on a σ-finite measure space (Ω, Σ, μ). By $\bar{L}_0$ we mean L_0 together with the constant function $+\infty$ and by $\mathbb{R}^+$ we mean $(0, \infty)$ together with the Lebesgue measure.

A function lattice E is understood to be a quasi-normed space in $L_0(\Omega)$ which satisfies that if $|f| \leqslant |g|$ a.e. and $g \in E$ then $f \in E$ and $\|f\| \leqslant \|g\|$.

We will consider an arbitrary nonempty cone P of $L_0(\Omega)$, i.e., $P + P \subset P$ and $\mathbb{R}^+ \cdot P \subset P$, and we shall assume that the functions in P take nonnegative values. To this cone, we associate the sublinear operator $\tilde{P}: L_0 \to \bar{L}_0$ defined as

$$\tilde{P}f := \inf\{g \in P: |f| \leqslant g\},$$

where $\tilde{P}f = +\infty$ when no $g \in P$ majorizes $|f|$ and we assume that $\tilde{P}f$ is measurable. For a function lattice E, we define

$$E^P := \{f \in L_0: \tilde{P}f \in E\},$$

which will be a function lattice under the quasi-norm

$$\|f\|_{E^P} := \|\tilde{P}f\|_E.$$

Clearly, we have $E^P \hookrightarrow E$, and if $\tilde{P}$ is bounded from E into E, then $E = E^P$. Set $\bar{E}^P := (E_0^P, ..., E_n^P)$, and throughout the paper we assume, when working with $\bar{E}^P$, that

$$\tilde{P}f \in P \qquad \text{for all} \quad f \in \bigcup E_i^P. \tag{2}$$

We remark that when working with the cone of concave or decreasing functions this hypothesis is always satisfied. But since this is not the case for a general cone, we need to assume it to be able to formulate our results in its whole generality.

Let us define, for $x, y \in \Omega$,

$$h^P(x, y) := \inf \left\{ \frac{f(y)}{f(x)} : f \in P, f(x) \neq 0 \right\}.$$

In what follows the function h^P will be of crucial importance. In our cases, we have that $h^P(x, \cdot)$ is measurable for almost all x and that $g_j(x) := \|h^P(x, \cdot)\|_{E_j}$ is measurable. In the general case we have to assume this.

For our purpose, let us write C for the cone of positive concave functions on $\mathbb{R}^+$ and D for the cone of positive decreasing functions on $\mathbb{R}^+$. We have that

$$h^C(x, y) = \min\left(1, \frac{y}{x}\right) \qquad \text{and} \qquad h^D(x, y) = \chi_{(0, x]}(y),$$

and we recall that if E is rearrangement invariant, then

$$g(x) := \|h^D(x, \cdot)\|_E$$

is the fundamental function of E.

We also need to recall the definition of the K-functional for an $(n+1)$-tuple of spaces $\bar{X} := (X_0, ..., X_n)$, all linearly embedded in a vector space $\mathscr{X}$. For $a \in \Sigma(\bar{X}) := X_0 + \cdots + X_n$ and $\bar{t} := (t_0, ..., t_n)$, $t_i > 0$, the K-functional is defined as

$$K(\bar{t}, a; \bar{X}) = \inf \left\{ \sum_{i=0}^{n} t_i \|a_i\|_{X_i} : a = \sum_{i=0}^{n} a_i, a_i \in X_i \right\}.$$

Throughout, the convention that $\|f\|_X = +\infty$ for $f \notin X$ will be used.

We write $f \lesssim g$ or $g \gtrsim f$ if there exists a strictly positive constant M such that for all x, $f(x) \leqslant Mg(x)$. If $f \lesssim g$ and $g \lesssim f$, the functions are said to be equivalent and we write $f \approx g$.

The paper is organized as follows. Section 2 contains the main results of this work, namely the introduction and investigation of the S-divisibility property. In particular, Holmstedt's formula is extended to this setting. In Section 3, we apply these results to study the K-functional for K-, rearrangement invariant, some symmetric, Lorentz, and weighted l_p spaces.

For those readers which are only interested in interpolation of couples we suggest reading the text in that context and to skip Sections 3.1 and 3.2.

2. S-DIVISIBILITY PROPERTY AND SOME OF ITS CONSEQUENCES

In this section, we will prove a Holmstedt-type formula which generalizes results of Holmstedt [15], Brudnyi and Krugljak [8], Nilsson [25], and Asekritova [2].

Let P be a cone and S an operator of the type

$$S\colon \mathscr{U} \to P,$$

where $\mathscr{U}$ is an additive group. Let us assume that S satisfies the sublinearity property

$$S(a_0 + \cdots + a_n) \leqslant T(S(a_0)) + \cdots + T(S(a_n)),$$

for some operator $T\colon P \to P$.

Given a lattice E, let us define the space $\mathscr{U}_{E;\,S}$ as

$$\mathscr{U}_{E;\,S} := \{a \in \mathscr{U}\colon Sa \in E\}, \ \|a\|_{\mathscr{U}_{E;\,S}} := \|Sa\|_E.$$

Note that $\|\cdot\|_{\mathscr{U}_{E;\,S}}$ need not be a norm nor $\mathscr{U}_{E;\,S}$ need even be a group. In particular, if $\mathscr{U} = \sum(\bar{A})$ and $(Sa)(t) = K(t, a; \bar{A})$, then $\mathscr{U}_{E;\,S}$ is the usual K-space and if $\mathscr{U} = L_0(\Omega)$ and $Sf = f^*$, $\mathscr{U}_{E;\,S} = E^*$ as defined in Section 1.

Then, in order to derive an expression of the K-functional for the family $\mathscr{U}_{\bar{E};\,S} := (\mathscr{U}_{E_0;\,S}, ..., \mathscr{U}_{E_n;\,S})$, where $\bar{E} := (E_0, ..., E_n)$ is a compatible tuple of lattices, we need to extend the notion of K-divisibility as follows:

Definition 1. Let $S\colon \mathscr{U} \to P$ be defined as above. Then $\mathscr{U}$ is said to be S-divisible with respect to P and the $(n+1)$-tuple of lattices $\bar{E}$ if, for every $a \in \mathscr{U}$ and $f_0, ..., f_n \in P$, for which

$$Sa \leqslant f_0 + \cdots + f_n,$$

there exists $a_i \in \mathscr{U}$ such that

$$a = a_0 + \cdots + a_n,$$

and

$$\|Sa_i\|_{E_i} \leqslant M \|f_i\|_{E_i},$$

for a constant M independent on a and f_i.

Let us mention some known examples of S-divisibility.

(1) *K-divisibility* (see [8]). $\bar{A} = (A_0, ..., A_m)$ is K-divisible if, for every a and $f_0, ..., f_n \in C$, such that

$$K(\cdot, a; \bar{A}) \leqslant f_0 + \cdots + f_n,$$

there exists $a_i \in \sum(\bar{A})$ satisfying

$$a = a_0 + \cdots + a_n, \qquad \text{and} \qquad K(\cdot, a_i; \bar{A}) \lesssim f_i.$$

Hence, if $\bar{A}$ is K-divisible and S is the K-functional, we find that $\mathscr{U} = \sum(\bar{A})$ is S-divisible with respect to C and all $(n+1)$-tuples of lattices on $\mathbb{R}^m_+$.

(2) *Weak K-divisibility* (see [3]). $\bar{A} = (A_0, ..., A_n)$ is said to be weak K-divisible if, for every a and every $f_0, ..., f_n \in C'$, where C' is the cone of positive concave functions on $\mathbb{R}^m_+$, such that

$$K(\cdot, a; \bar{A}) \leqslant f_0 + \cdots + f_n,$$

there exists $a_i \in \sum(\bar{A})$ satisfying

$$a = a_0 + \cdots + a_n, \qquad \text{and} \qquad K(\cdot, a_i; \bar{A}) \lesssim Rf_i,$$

where

$$R(f)(t_1, ..., t_m) := \int_{\mathbb{R}^m_+} \min\left\{1, \frac{t_1}{s_1}, ..., \frac{t_m}{s_m}\right\} f(s_1, ..., s_m) \frac{ds_1}{s_1} \cdots \frac{ds_m}{s_m}.$$

Hence, if $\bar{A}$ is weak K-divisible and S is the K-functional, we see that $\mathscr{U} = \sum(\bar{A})$ is S-divisible with respect to C and the $(n+1)$-tuples consisting of lattices on which R is bounded.

In [4] it was implicitly shown that all $(n+1)$-tuples of functional Banach lattices possesses this property of weak K-divisibility for all $a \in \sum(\bar{A})$ such that

$$[R^2 K(\cdot, a; \bar{A})](1) < +\infty,$$

if R is replaced with R^2.

(3) *E-divisibility* (see [25]). Let $\bar{A}=(A_0, A_1)$ be a couple of p-normed complete abelian groups and let us consider the E-functional

$$E(t, a; \bar{A}) := \inf\{\|a-b\|_{A_0}: \|b\|_{A_1} \leqslant t\}.$$

Let $\sum_E(\bar{A})$ denote the subset of $\sum(\bar{A})$ for which $E(\cdot, a; \bar{A})$ is finite everywhere.

Then, $\bar{A}$ is said to be E-divisible if, for every a and every $f_0, f_1 \in D' := \{f \in D: f(t) \to 0 \text{ as } t \to \infty\}$, such that

$$E(\cdot, a; \bar{A}) \leqslant f_0 + f_1,$$

there exists $a_i \in \sum_A(\bar{A})$ satisfying

$$a = a_0 + a_1, \qquad \text{and} \qquad E(\cdot, a_i; \bar{A}) \lesssim f_i(\cdot/c),$$

for a constant c. In [25], it is proved that all couples of this type are E-divisible. Hence, if S is the E-functional, $\mathscr{U} = \sum_E(\bar{A})$ is S-divisible with respect to the cone D' and all couples of lattices on which the dilation is bounded.

More examples of S-divisibility can be found in the next section.

First we prove the following Holmstedt type formula which generalizes results of Holmstedt [15], Brudyi and Krugljak [8], Nilsson [25] and Asekritova [2].

THEOREM 2. *Let $\mathscr{U}$ be S-divisible with respect to P and $\bar{E}$. If $\|Tf\|_{E_i} \lesssim \|f\|_{E_i}$ for all $f \in P \cap E_i$, then*

$$K(\bar{t}, a; \mathscr{U}_{\bar{E};S}) \approx K(\bar{t}, Sa; \bar{E}^P). \tag{3}$$

Proof. Let $a = \sum_{i=0}^n a_i$. Then $Sa \leqslant \sum_{i=0}^n TSa_i$ and, hence,

$$K(\bar{t}, Sa; \bar{E}^P) \leqslant K\left(\bar{t}, \sum_{i=0}^n TSa_i; \bar{E}^P\right) \leqslant \sum_{i=0}^n t_i \|TSa_i\|_{E_i^P} = \sum_{i=0}^n t_i \|TSa_i\|_{E_i}$$

$$\lesssim \sum_{i=0}^n t_i \|Sa_i\|_{E_i} = \sum_{i=0}^n t_i \|a_i\|_{\mathscr{U}_{E_i;S}}.$$

Conversely, let $Sa = f_0 + \cdots + f_n$. Therefore $Sa \leqslant \tilde{P}f_0 + \cdots + \tilde{P}f_n$ and hence, by the S-divisibility property, there exists a decomposition $a = \sum_{i=0}^n a_i$ and $g_i \in P$ such that $Sa_i \lesssim g_i$ and $\|g_i\|_{E_i} \lesssim \|\tilde{P}f_i\|_{E_i}$. Then we have that

$$K(\bar{t}, a; \mathscr{U}_{\bar{E};\,S}) \leqslant \sum_{i=0}^{n} t_i \, \|a_i\|_{\mathscr{U}_{E_i;\,S}} = \sum_{i=0}^{n} t_i \, \|Sa_i\|_{E_i} \lesssim \sum_{i=0}^{n} t_i \, \|g_i\|_{E_i}$$
$$\lesssim \sum_{i=0}^{n} t_i \, \|\tilde{P} f_i\|_{E_i} = \sum_{i=0}^{n} t_i \, \|f_i\|_{E_i^P},$$

and the proof is complete. ∎

By applying the quasi-norm of a function lattice G, we immediately get the following reiteration result:

$$(\mathscr{U}_{\bar{E};\,S})_{G;\,K} = \mathscr{U}_{(\bar{E}^P)_{G;\,K};\,S}.$$

If we now want to find an explicit formula for $K(\bar{t}, a; \mathscr{U}_{\bar{E};\,S})$, we can restrict ourselves to get such a formula for $K(\bar{t}, f; \bar{E}^P)$ with f not an arbitrary function but a function in the cone P.

For this purpose it will be enough to find an "almost optimal" decomposition for f in the following sense: For a given $c > 0$, we say that $f = f_0 + \cdots + f_n$ is an almost optimal decomposition if

$$\sum_{i=0}^{n} t_i \, \|f_i\|_{E_i} \leqslant cK(\bar{t}, f, \bar{E}).$$

We recall that the original Holmstedt's formula expresses not only a connection between

$$K(t, a; \bar{A}_{\theta_0,\,p_0}, \bar{A}_{\theta_1,\,p_1}) \qquad \text{and} \qquad K(t, K(\,\cdot\,, a; \bar{A}); L_{p_0}^{\theta_0}, L_{p_1}^{\theta_1})$$

but also an almost optimal decomposition of the type

$$K(\,\cdot\,, a; \bar{A}) = K(\,\cdot\,, a; \bar{A})\, \chi_{(0,\,b_t)} + K(\,\cdot\,, a; \bar{A})\, \chi_{[b_t,\,\infty)}$$

is found. Our next purpose is to deal with this type of almost optimal decompositions for $f \in P$.

THEOREM 3. *Let $\bar{E}$ be an $(n+1)$-tuple of lattices which satisfies that, for $f \in P$,*

$$K(\bar{t}, f; \bar{E}) \approx \inf \left\{ \sum_{i=0}^{n} t_i \, \|f_i\|_{E_i} : f \leqslant f_0 + \cdots + f_n,\, f_i \in P \right\}. \tag{4}$$

If there exist measurable sets $B_i(\bar{t})$ such that $\bigcup_{i=0}^{n} B_i(\bar{t}) = \Omega$ and

$$\sup_{\bar{t}} \frac{t_i}{t_j} \, \|\chi_{B_i(\bar{t})}/g_j\|_{E_i} < \infty, \qquad i \neq j,$$

where $g_j(x) := \|h^P(x, \cdot)\|_{E_j}$, *then*

$$K(\bar{t}, f; \bar{E}) \approx \sum_{i=0}^{n} t_i \|f\chi_{B_i(\bar{t})}\|_{E_i},$$

for all $f \in P$.

Proof. Obviously $\leqslant$ holds. Conversely, let us take $f \leqslant f_0 + \cdots + f_n$, $f_i \in P$ arbitrary, Then,

$$\sum_{i=0}^{n} t_i \|f\chi_{B_i(\bar{t})}\|_{E_i} \leqslant \sum_{i=0}^{n} t_i \|(f_0 + \cdots + f_n)\chi_{B_i(\bar{t})}\|_{E_i} \lesssim \sum_{i=0}^{n} t_i \|f_i\|_{E_i} + \sum_{i \neq j} q_{ij},$$

where $q_{ij} := t_i \|f_j \chi_{B_i(\bar{t})}\|_{E_i}$. The proof will follow if we prove that $q_{ij} \lesssim t_j \|f_j\|_{E_j}$. In fact, since $f_j \in P$, we see that $f_j(x)\, h^P(x, y) \leqslant f_j(y)$ and, hence, $f_j(x)\, g_j(x) \leqslant \|f_j\|_{E_j}$. Therefore

$$q_{ij} \leqslant t_i \|\chi_{B_i(\bar{t})} \|f_j\|_{E_j} / g_j\|_{E_i} = t_j \|f_j\|_{E_j} \frac{t_i}{t_j} \|\chi_{B_i(\bar{t})}/g_j\|_{E_i} \lesssim t_j \|f_j\|_{E_j},$$

and the proof is complete. ∎

Combining the information from Theorems 2 and 3 we also obtain the following result.

THEOREM 4. *Let* $\mathscr{U}$ *be S-divisible with respect to P and* $\bar{E}$, *and assume that* $\|Tf\|_{E_i} \lesssim \|f\|_{E_i}$ *for all* $f \in P \cap E_i$.
If there exist measurable sets $B_i(\bar{t})$ *such that* $\bigcup_{i=0}^{n} B_i(\bar{t}) = \Omega$ *and*

$$\sup_{\bar{t}} \frac{t_i}{t_j} \|\chi_{B_i(\bar{t})}/g_j\|_{E_i^P} < \infty, \qquad i \neq j, \tag{5}$$

where $g_j(x) = \|h^P(x, \cdot)\|_{E_j^P}$, *then*

$$K(\bar{t}, a; \mathscr{U}_{\bar{E};\, S}) \approx \sum_{i=0}^{n} t_i \|Sa\chi_{B_i(\bar{t})}\|_{E_i^P}.$$

Proof. According to Theorems 2 and 3, we only have to check that condition (4) is satisfied, and this follows immediately by the sublinearity of $\tilde{P}$ and the lattice property of the K-functional together with our assumption (2). ∎

3. APPLICATIONS AND FURTHER RESULTS

In this section we present some applications and further results related to the theory developed in the previous section.

3.1. *On Asekritova's Holmstedt Type Formula*

Let C' be the cone of positive concave functions on $\mathbb{R}^m_+$. Then one can easily see that

$$h^{C'}(x, y) = \min\left\{1, \frac{y_1}{x_1}, ..., \frac{y_m}{x_m}\right\}.$$

Moreover, it follows from [8] that if $\bar{L} := (L_\infty, L^{t_1}_\infty, ..., L^{t_m}_\infty)$ where $L^{t_i}_\infty = L_\infty(w)$ with $w(t) = 1/t_i$, we have that

$$\|f\|_{E^{C'}} = \|f\|_{\bar{L}_{E;\,K}},$$

and, hence, we can obtain as an immediate application of Theorem 4, the following generalized Holmstedt formula stated in [2]:

THEOREM 5. *Let $\bar{A} := (A_0, ..., A_m)$ be a K-divisible $(m+1)$-tuple of Banach spaces and $\bar{E} := (E_0, ..., E_n)$ an $(n+1)$-tuple of Banach lattices on $\mathbb{R}^m_+$ such that* $\min(1, t_1, ..., t_m) \in E_i$.

If there exist measurable sets $B_i(\bar{t})$ such that $\bigcup_{i=0}^n B_i(\bar{t}) = \mathbb{R}^m_+$ and

$$\sup_{\bar{t}} \frac{t_i}{t_j} \|\chi_{B_i(\bar{t})}/g_j\|_{\bar{L}_{E;\,K}} < \infty, \qquad i \neq j,$$

then

$$K(\bar{t}, a; \bar{A}_{E_0;\,K}, ..., \bar{A}_{E_n;\,K}) \approx \sum_{i=0}^n t_i \|K(\cdot, a; \bar{A})\, \chi_{B_i(\bar{t})}\|_{\bar{L}_{E;\,K}}.$$

Moreover, using Theorem 4, we can also derive the following description of the K-functional for the $(n+1)$-tuple

$$((X, Y)_{\theta_0,\, q_0}, ..., (X, Y)_{\theta_n,\, q_n})$$

(see also [2]).

THEOREM 6. *Let $0 \leqslant \theta_0 < \cdots < \theta_n \leqslant 1$ and $1 \leqslant q_i \leqslant \infty$ ($q_0 = \infty$ if $\theta_0 = 0$ and $q_n = \infty$ if $\theta_n = 1$). Then*

$$K(\bar{t}, a; (X, Y)_{\theta_0,\, q_0}, ..., (X, Y)_{\theta_n,\, q_n}) \approx \sum_{i=0}^n t_i \|K(\cdot, a; X, Y)\, \chi_{B_i(\bar{t})}\|_{L^{\theta_i}_{q_i}}, \quad (6)$$

where

$$B_i(\bar{t}) := \left\{ s \in \mathbb{R}^+ : s^{\theta_j - \theta_i} \leqslant \frac{t_j}{t_i},\ j = 0, ..., n \right\}.$$

Moreover, if $\theta_0 = 0$ (*or* $\theta_n = 1$), *then the first* (*resp. the last*) *term in the sum can be removed but not both at the same time.*

Proof. First we note that

$$g_i(s) := \|h^C(s, \cdot)\|_{(L^{\theta_i}_{q_i})^C} = \|h^C(s, \cdot)\|_{L^{\theta_i}_{q_i}} \approx s^{-\theta_i},$$

and, hence,

$$\|\chi_{[x_i, y_i]}/g_j\|_{(L^{\theta_i}_{q_i})^C} \lesssim \begin{cases} y_i^{\theta_j - \theta_i}, & i < j, \\ x_i^{\theta_j - \theta_i}, & i > j. \end{cases}$$

Therefore, if we take $B_i = [x_i, y_i]$ with

$$x_i := \max_{j<i} \{(t_j/t_i)^{1/(\theta_j - \theta_i)}\} \qquad \text{and} \qquad y_i := \min_{j>i} \{(t_j/t_i)^{1/(\theta_j - \theta_i)}\}$$

($B_i = \phi$ if $x_i > y_i$), we find that condition (5) holds and since, by induction over n, one can easily see that $\bigcup_{i=0}^n B_i = \mathbb{R}^+$, we have that the hypotheses of Theorem 4 are satisfied!

We have then to show that

$$\sum_{i=0}^n t_i \|K(\cdot, a; X, Y)\, \chi_{B_i(\bar{t})}\|_{(L^{\theta_i}_{q_i})^C} \approx \sum_{i=0}^n t_i \|K(\cdot, a; X, Y)\, \chi_{B_i(\bar{t})}\|_{L^{\theta_i}_{q_i}}.$$

Of course only the approximative inequality $\lesssim$ has to be considered. We have that

$$\|K(\cdot, a; X, Y)\, \chi_{B_i(\bar{t})}\|_{(L^{\theta_i}_{q_i})^C} \approx x_i^{-\theta_i} K(x_i, a; X, Y) + y_i^{-\theta_i} K(y_i, a; X, Y) + \|K(\cdot, a; X, Y)\, \chi_{B_i(\bar{t})}\|_{L^{\theta_i}_{q_i}},$$

and, hence, we must prove that

$$t_i x_i^{-\theta_i} K(x_i, a; X, Y) \lesssim \sum_{i=0}^n t_i \|K(\cdot, a; X, Y)\, \chi_{B_i(\bar{t})}\|_{L^{\theta_i}_{q_i}},$$

and

$$t_i y_i^{-\theta_i} K(y_i, a; X, Y) \lesssim \sum_{i=0}^n t_i \|K(\cdot, a; X, Y)\, \chi_{B_i(\bar{t})}\|_{L^{\theta_i}_{q_i}}.$$

For $q_i = \infty$, this is clear and otherwise we consider two cases:

(i) If $2x_i < y_i$, then, by using the concavity property of the K-functional, we find that

$$\begin{aligned}\int_{x_i}^{y_i} [s^{-\theta_i} K(s, a)]^{q_i} \frac{ds}{s} &\geqslant K(x_i, a)^{q_i} \int_{x_i}^{y_i} s^{-\theta_i q_i - 1}\, ds \\ &\approx K(x_i, a)^{q_i} [x_i^{-\theta_i q_i} - y_i^{-\theta_i q_i}] \\ &\geqslant K(x_i, a)^{q_i} [x_i^{-\theta_i q_i} - (2x_i)^{-\theta_i q_i}] \approx [x_i^{-\theta_i} K(x_i, a)]^{q_i},\end{aligned}$$

and similarly for the other estimate.

(ii) Let us now assume that $2x_i \geqslant y_i$, and let B_j be the interval left of B_i, i.e., $y_j = x_i$. Since $x_i \in B_j \cap B_i$, we have $x_i^{\theta_j - \theta_i} = t_j / t_i$ and hence

$$t_i y_i^{-\theta_i} K(y_i, a; X, Y) \approx t_i x_i^{-\theta_i} K(x_i, a; X, Y) = t_j y_j^{-\theta_j} K(y_j, a; X, Y).$$

In this way we have moved one interval to the left. If also $2x_j \geqslant y_j$ we repeat the same procedure again and since $2x_0 = 0 < y_0$ this process will stop after finite many steps.

To prove the last part of the theorem, let us assume that $\theta_n = 1$ (the case $\theta_0 = 0$ follows similarly). We want to prove that

$$t_n x_n^{-1} K(x_n, a; X, Y) \lesssim \sum_{i=0}^{n-1} t_i \|K(\cdot, a; X, Y)\, \chi_{B_i(\bar{t})}\|_{L_{q_i}^{\theta_i}}. \tag{7}$$

Let B_{i_k}, $k = 0, ..., l$, be the nonempty sets B_i. Now, by the concavity property of the K-functional and the fact that $t_i x_i^{-\theta_i} = t_j y_j^{-\theta_j}$ if B_j is the interval left to B_i, we have that

$$\begin{aligned}\sum_{i=0}^{n-1} t_i \|K(\cdot, a; X, Y)\, \chi_{B_i(\bar{t})}\|_{L_{q_i}^{\theta_i}} &\gtrsim x_n^{-1} K(x_n, a; X, Y) \sum_{k=0}^{l-1} t_{i_k} (y_{i_k}^{1-\theta_{i_k}} - x_{i_k}^{1-\theta_{i_k}}) \\ &= x_n^{-1} K(x_n, a; X, Y)\, t_{i_{l-1}} y_{i_{l-1}}^{1-\theta_{i_{l-1}}} \\ &= t_n x_n^{-1} K(x_n, a; X, Y),\end{aligned}$$

and (7) follows. ∎

Let us now consider the finite family

$$(X, (X, Y)_{\theta_1, q_1}, ..., (X, Y)_{\theta_{n-1}, q_{n-1}}, Y).$$

A formula for the K-functional for this family can be obtained as follows: First we have to check that

$$K(\bar{t}, a; X, A_1, ..., A_{n-1}, Y) = K(\bar{t}, a; X^c, A_1, ..., A_{n-1}, Y^c),$$

where X^c and Y^c are the Gagliardo completions of X and Y with respect to $X+Y$. This follows similarly as for the case with pairs, (see e.g. [6]). Now it only remains to recall that $X^c = (X, Y)_{0,\infty}$ and $Y^c = (X, Y)_{1,\infty}$, and the previous result applies.

3.2. *The K-Functional for Rearrangement Invariant Spaces*

As usual, by rearrangement invariant spaces (r.i), we mean quasi-normed lattices E defined on $\mathbb{R}^+$ which are complete and satisfying $\|f\|_E = \|f^*\|_E$. Let us consider $(n+1)$-tuples $\bar{E}$ of r.i. and let us recall that, as it was mentioned in the introduction, the fundamental function g_i is defined by

$$g_i(x) = \|\chi_{(0,x]}\|_{E_i} = \|h^D(x, \cdot)\|_{E_i}.$$

It is known (see [17, 11]) that for a couple $\bar{E}$ of r.i.

$$K(t, f; \bar{E}) \approx K(t, f^*, \bar{E}),$$

and this equivalence can easily be extended to the case of a finite family of r.i. Moreover, it follows from a result in [16] that the dilation operator is bounded on these spaces.

Using these facts together with our Theorem 3, we are able to extend some of the results in [13, 19 and 22].

THEOREM 7. *Let $\bar{E} := (E_0, ..., E_n)$ be an $(n+1)$-tuple of r.i. and let g_i be the fundamental function of E_i, $i = 0, ..., n$. If there exist measurable sets $B_i(\bar{t})$ such that $\bigcup_{i=0}^n B_i(\bar{t}) = \mathbb{R}^+$ and*

$$\sup_{\bar{t}} \frac{t_i}{t_j} \|\chi_{B_i(\bar{t})}/g_j\|_{E_i} < \infty, \qquad i \neq j, \tag{8}$$

then

$$K(\bar{t}, f; \bar{E}) \approx \sum_{i=0}^n t_i \|f^* \chi_{B_i(\bar{t})}\|_{E_i}.$$

Proof. In view of our Theorem 3, we only need to prove that

$$K(\bar{t}, f^*; \bar{E}) \approx \inf\left\{\sum_{i=0}^n t_i \|f_i\|_{E_i} : f^* \leqslant f_0 + \cdots + f_n, f_i \in D\right\}. \tag{9}$$

Let $f^* \leqslant f_0 + \cdots + f_n$, $f_i \in D$. Then

$$K(\bar{t}, f^*; \bar{E}) \leqslant K\left(\bar{t}, \sum_{i=0}^{n} f_i; \bar{E}\right) \leqslant \sum_{i=0}^{n} t_i \, \|f_i\|_{E_i},$$

and the inequality $\leqslant$ in (9) follows. Conversely, if $f^* = f_0 + \cdots + f_n$, then we have that

$$f^* \leqslant f_0^*(\cdot/n) + \cdots + f_n^*(\cdot/n)$$

and, hence, the right-hand side of (9) is not bigger than

$$\sum_{i=0}^{n} t_i \, \|f_i^*(\cdot/n)\|_{E_i} \lesssim \sum_{i=0}^{n} t_i \, \|f_i^*\|_{E_i} = \sum_{i=0}^{n} t_i \, \|f_i\|_{E_i}.$$

By taking the infimum over all $f^* = f_0 + \cdots + f_n$, we have also proved the approximative inequality $\gtrsim$ in (9) and we are done. ∎

As an application to the setting of $L_p(\mathbb{R}^+)$ spaces, we get the following extension of the Holmstedt formula (see also [13]).

COROLLARY 8. *Let* $0 < p_0 < \cdots < p_n \leqslant \infty$. *Then*

$$K(\bar{t}, f; L_{p_0}, ..., L_{p_n}) \approx \sum_{i=0}^{n} t_i \, \|f^* \chi_{B_i(\bar{t})}\|_{L_{p_i}},$$

where

$$B_i(\bar{t}) := \left\{ s \in \mathbb{R}^+ : s^{1/p_i - 1/p_j} \leqslant \frac{t_j}{t_i}, \; j = 0, ..., n \right\}.$$

Proof. According to Theorem 7, we only have to check that condition (8) is satisfied. Let us take $B_i(\bar{t}) := [x_i, y_i]$ with

$$x_i = \max_{j < i} \{(t_j/t_i)^{(p_i p_j)/(p_j - p_i)}\} \qquad \text{and} \qquad y_i = \min_{i < j} \{(t_j/t_i)^{(p_i p_j)/(p_j - p_i)}\},$$

$B_i = \phi$ if $x_i > y_i$. Then, by induction over n, one can easily see that $\bigcup_{i=0}^{n} B_i(\bar{t}) = \mathbb{R}^+$. Moreover, since $g_i(s) = s^{1/p_i}$, we find that

$$\frac{t_i}{t_j} \, \|\chi_{B_i(\bar{t})}/g_j\|_{L_{p_i}} \lesssim \frac{t_i}{t_j} \, x_i^{1/p_i - 1/p_j}, \qquad \text{for} \quad j < i,$$

and

$$\frac{t_i}{t_j}\,\|\chi_{B_i(\bar{t})}/g_j\|_{L_{p_i}} \lesssim \frac{t_i}{t_j}\, y_i^{1/p_i - 1/p_j}, \qquad \text{for} \quad i<j,$$

and hence (8) holds. ▮

3.3. *The K-Functional for Some Symmetric Spaces*

For a given lattice E on which the dilation operator is bounded, we let E^* be the symmetric space of all measurable functions f on Ω so that $f^* \in E$ under the quasi-norm

$$\|f\|_{E^*} = \|f^*\|_E.$$

In order to be able to deal with an expression for $K(t, f; E_0^*, E_1^*)$, we will prove the following lemma of independent interest. In particular it shows that $L_0(\Omega)$ is S-divisible with respect to the cone D and all pairs of lattices $\bar{E}$ on $\mathbb{R}^+$, where $Sf = f^*$.

LEMMA 9. *Let $f\colon \Omega \to [0, \infty)$ be a measurable function and let us assume that there exist two decreasing functions, f_0 and f_1, so that*

$$f^* \leqslant f_0 + f_1.$$

Then, there exist two measurable functions $g_j\colon \Omega \to [0, \infty)$, $j=0, 1$, such that $f = g_0 + g_1$ and $g_j^* \leqslant f_j$, $j=0, 1$.

Proof. First we observe that using the decomposition property of the cone of decreasing functions (see [9] and [12]) we can assume, without loss of generality, that $f^* = f_0 + f_1$.

Let us first assume that the measure space (Ω, Σ, μ) is resonant and let $\alpha = \lim_{t \to \infty} f^*(t)$. If $\alpha = 0$ the result follows from the fact (see [6] and [27]) that there exists a measure preserving transformation σ such that $f = f^* \circ \sigma$ and hence taking $g_j = f_j \circ \sigma$ and we are done.

Now, if $\alpha > 0$, we have that $\mu(\Omega) = \infty$ and, $\alpha = \alpha_0 + \alpha_1$ where $\alpha_j = \lim_{t \to \infty} f_j(t)$. Let $E = \{x \in \Omega\colon f(x) > \alpha\}$ and let $F = (f - \alpha)\,\chi_E$. Then

$$F^* = f^* - \alpha = (f_0 - \alpha_0) + (f_1 - \alpha_1) = h_0 + h_1.$$

Since $\lim_{t \to \infty} F^*(t) = 0$, we have that there exist G_j so that $F = G_0 + G_1$ and $G_j^* \leqslant h_j$ for $j = 0, 1$. Set

$$g_j = (G_j + \alpha_j)\,\chi_E + \frac{\alpha_j}{\alpha}\, f\chi_{\Omega \setminus E}.$$

Then $g_0+g_1=(F+\alpha)\,\chi_E+f\chi_{\Omega\setminus E}=f$. Let us now show that $g_j^*\leqslant f_j$; equivalently $\lambda_{g_j}\leqslant\lambda_{f_j}$. If $0<s<\alpha_j$ we have that $\lambda_{f_j}(s)=\infty$ and

$$\begin{aligned}\mu\{x\in\Omega\colon g_j(x)>s\}&=\mu(E)+\mu\left\{x\in\Omega\setminus E\colon f(x)>\frac{\alpha s}{\alpha_j}\right\}\\&=\mu\left\{x\in\Omega\colon f(x)>\frac{\alpha s}{\alpha_j}\right\}=\infty,\end{aligned}$$

and, if $s>\alpha_j$

$$\begin{aligned}\mu\{x\in\Omega\colon g_j(x)>s\}&=\mu\{x\in E\colon G_j(x)+\alpha_j>s\}+\mu\left\{x\in\Omega\setminus E\colon f(x)>\frac{\alpha s}{\alpha_j}\right\}\\&=\lambda_{G_j}(s-\alpha_j)\leqslant\lambda_{h_j}(s-\alpha_j)=\lambda_{f_j}(s).\end{aligned}$$

Let us now remark that, by the construction, one can easily see that if $c>0$ and $E_c=\{x\in\Omega\colon f(x)=c\}$ has positive measure, then g_j are constant on E_c.

Finally, for a general σ-finite measure space, we use the method of retracts (see [6]), which enables us to embed the space (Ω,Σ,μ) into a nonatomic (and hence resonant) measure space $(\bar{\Omega},\bar{\Sigma},\bar{\mu})$ in the following way: Let us write

$$\Omega=\Omega_0\cup\left(\bigcup_n A_n\right),$$

where Ω_0 is nonatomic and each A_n is an atom of finite positive measure. Now we can consider $\bar{\Omega}=\Omega_0\cup(\bigcup_n I_n)$ where I_n are pairwise disjoint intervals with $|I_n|=\mu(A_n)$ and I_n also disjoint with Ω_0. Then

$$\bar{\mu}(E)=\mu(E\cap\Omega_0)+\sum_n|E\cap I_n|,$$

defines a nonatomic measure on $\bar{\Omega}$. Moreover, for a given function f on (Ω,Σ,μ) the function $\varepsilon(f)$ on $(\bar{\Omega},\bar{\Sigma},\bar{\mu})$, defined by $\varepsilon(f)=f$ on Ω_0 and on I_n it is defined as the constant value f on each A_n, satisfies $\varepsilon(f)^*_{\bar{\mu}}=f^*_{\mu}$.

Using the previous argument, we can find h_j on $(\bar{\Omega},\bar{\Sigma},\bar{\mu})$ so that $h_j^*\leqslant f_j$ and $\varepsilon(f)=h_0+h_1$. Since $\varepsilon(f)$ is constant on I_n, we obtain that h_j are also constant on it and therefore $f=g_0+g_1$, where $g_j=h_j$ on Ω_0 and on A_n g_j equals the constant value of h_j on I_n. Since $g_j^*\leqslant f_j$, we are done. ∎

Therefore, by Theorem 2 and the previous lemma we have that

$$K(t,f;E_0^*,E_1^*)\approx K(t,f^*;E_0^D,E_1^D),$$

which, under some suitable assumption on $\bar{E}$, leads us to the description

$$K(t, f; E_0^*, E_1^*) \approx K(t, f^*; E_0, E_1)$$

(see [25]). In the particular case that E_0 and E_1 are rearrangement invariant Banach space in $(0, \infty)$, the description of the above K-functional has recently been given in [5].

As a first application of the above lemma we have

THEOREM 10. *If the dilation operator is bounded on E, then we have*

$$K(\varphi(t), f, E^*, L_\infty) \approx \|f^*\chi_{(0,t)}\|_E,$$

where $\varphi(t) = \|\chi_{(0,t)}\|_E$.

Proof. It holds that

$$\begin{aligned} K(\varphi(t), f, E^*, L_\infty) &= \inf_{\lambda>0} \{\|(|f|-\lambda)_+\|_{E^*} + \lambda\varphi(t)\} \\ &= \inf_{\lambda>0} \{\|(f^*-\lambda)_+\|_E + \lambda\varphi(t)\} \\ &\geqslant \inf_{\lambda>0} \sup_{\delta>0} \{\|(f^*-\lambda)_+ \chi_{(0,\delta)}\|_E + \lambda\varphi(t)\} \\ &\gtrsim \inf_{\lambda>0} \sup_{\delta>0} \|f^*\chi_{(0,\delta)}\|_E - \lambda\varphi(\delta) + \lambda\varphi(t) \\ &\geqslant \|f^*\chi_{(0,t)}\|_E. \end{aligned}$$

Conversely, according to Theorem 2, it yields that

$$K(\varphi(t), f, E^*, L_\infty) \approx K(\varphi(t), f^*, E^D, (L_\infty)^D).$$

Hence,

$$\begin{aligned} K(\varphi(t), f, E^*, L_\infty) &\lesssim \|f^*\chi_{(0,t)}\|_{E^D} + \varphi(t)\ \|f^*\chi_{[t,\infty)}\|_{(L_\infty)^D} \\ &= \|f^*\chi_{(0,t)}\|_E + \varphi(t)\ \|f^*\chi_{[t,\infty)}\|_{L_\infty}. \end{aligned}$$

Thus

$$K(\varphi(t), f, E^*, L_\infty) \lesssim \|f^*\chi_{(0,t)}\|_E + \varphi(t)\ f^*(t) \lesssim \|f^*\chi_{(0,t)}\|_E,$$

and the proof is complete. ∎

We now turn to weighted Lorentz spaces. Let $0<p<\infty$, let w be a positive, locally integrable function and

$$W(t):=\int_0^t w(x)\,dx.$$

The weighted Lorentz space $\Lambda^p(w)$ is defined to be the set of all measurable functions such that

$$\|f\|_{\Lambda^p(w)}:=\left(\int_0^\infty f^*(x)^p\,w(x)\,dx\right)^{1/p}<+\infty.$$

We assume that W satisfies the Δ_2-condition and hence $\|f\|_{\Lambda^p(w)}$ defines a quasi-norm, see [10].

Our aim is to give a formula for the K-functional

$$K(t, f; \Lambda^{p_0}(w_0), \Lambda^{p_1}(w_1)),$$

for $0<p_i<\infty$, by using the fact that $\Lambda^p(w)=L_0(\Omega)_{E;\,S}$ where $Sf=f^*$ and $E=L_p(w)$.

We have the following result.

THEOREM 11. *If, for every $t>0$, there exists $a_t\in[0,\infty]$ such that*

$$\left(\int_0^{a_t}\frac{w_0(s)}{W_1(s)^{p_0/p_1}}\,ds\right)^{1/p_0}\lesssim t, \qquad \textit{and} \tag{10}$$

$$\left(\frac{W_1(a_t)}{W_0(a_t)^{p_1/p_0}}+\int_{a_t}^\infty\frac{w_1(s)}{W_0(s)^{p_1/p_0}}\,ds\right)^{1/p_1}\lesssim\frac{1}{t}, \tag{11}$$

then

$$K(t, f; \Lambda^{p_0}(w_0), \Lambda^{p_1}(w_1)) \approx\left(\int_0^{a_t}f^*(s)^{p_0}\,w_0(s)\,ds\right)^{1/p_0}+t\left(\int_{a_t}^\infty f^*(s)^{p_1}\,w_1(s)\,ds\right)^{1/p_1} \tag{12}$$

for all $f\in\Lambda^{p_0}(w_0)+\Lambda^{p_1}(w_1)$.

Proof. According to Theorem 4, it is sufficient to prove that (5) holds with $B_0=(0,a_t)$ and $B_1=[a_t,\infty)$. In fact, we have that

$$g_i(x):=\|h^D(x,\cdot)\|_{(L_{p_i}(w_i))^D}=\|\chi_{(0,x]}\|_{L_{p_i}(w_i)}=W_i(x)^{1/p_i},$$

and, hence, by (10),

$$\sup_{t>0} \frac{1}{t} \|\chi_{(0,\, a_t)}/g_1\|_{(L_{p_0}(w_0))^D} = \sup_{t>0} \frac{1}{t} \left(\int_0^{a_t} \frac{w_0(s)}{W_1(s)^{p_0/p_1}}\, ds \right)^{1/p_0} < +\infty,$$

and, by (11),

$$\sup_{t>0} t \, \|\chi_{[a_t,\infty)}/g_0\|_{(L_{p_1}(w_1))^D} = \sup_{t>0} t \, \|\chi_{(0,a_t)}/g_0(a_t) + \chi_{[a_t,\infty)}/g_0\|_{L_{p_1}(w_1)}$$
$$= \sup_{t>0} t \left(\frac{W_1(a_t)}{W_0(a_t)^{p_1/p_0}} + \int_{a_t}^{\infty} \frac{w_1(s)}{W_0(s)^{p_1/p_0}}\, ds \right)^{1/p_1} < +\infty.$$

It now remains to prove that

$$\|f^* \chi_{B_0}\|_{(L_{p_0}(w_0))^D} + t\, \|f^* \chi_{B_1}\|_{(L_{p_1}(w_1))^D} \approx \|f^* \chi_{B_0}\|_{L_{p_0}(w_0)} + t\, \|f^* \chi_{B_1}\|_{L_{p_1}(w_1)},$$

i.e., we have to check that

$$tf^*(a_t)\, \|\chi_{(0,\, a_t)}\|_{L_{p_1}(w_1)} \lesssim \|f^* \chi_{B_0}\|_{L_{p_0}(w_0)} + t\, \|f^* \chi_{B_1}\|_{L_{p_1}(w_1)}.$$

But, in view of (11), we have that

$$\frac{W_1(a_t)^{1/p_1}}{W_0(a_t)^{1/p_0}} \lesssim \frac{1}{t},$$

which implies

$$tf^*(a_t)\, \|\chi_{(0,\, a_t)}\|_{L_{p_1}(w_1)} \lesssim \|f^* \chi_{B_0}\|_{L_{p_0}(w_0)},$$

and the proof is complete. ▮

We now look at two special cases of Theorem 11.

COROLLARY 12. *Formula* (12), *with* a_t *such that* $t = W_0(a_t)^{1/p_0}\, W_1(a_t)^{-1/p_1}$, *holds in the two following cases.*

(a) *If* $w_i(t) \lesssim W_i(t)/t$, *a.e., and there exists* $c > 0$ *so that*

$$\frac{W_0(t)^{1/p_0}}{W_1(t)^{1/p_1}}\, t^{-c}, \tag{13}$$

increases.

(b) *If there exist* $\alpha > 0$ *and* $\beta > 0$ *such that* $\alpha p_0 < \beta p_1$ *and* W_0^α / W_1^β *increases.*

Proof. (a) The above assumptions yield

$$\left(\int_0^{a_t} \frac{w_0(s)}{W_1(s)^{p_0/p_1}}\,ds\right)^{1/p_0} \lesssim \left(\int_0^{a_t} \frac{W_0(s)/s}{W_1(s)^{p_0/p_1}}\,ds\right)^{1/p_0}$$

$$= \left(\int_0^{a_t} \left[\frac{W_0(s)^{1/p_0}\,s^{-c}}{W_1(s)^{1/p_1}}\right]^{p_0} s^{cp_0}\,\frac{ds}{s}\right)^{1/p_0}$$

$$\lesssim \frac{W_0(a_t)^{1/p_0}}{W_1(a_t)^{1/p_1}},$$

and, similarly,

$$\left(\frac{W_1(a_t)}{W_0(a_t)^{p_1/p_0}} + \int_{a_t}^{\infty} \frac{w_1(s)}{W_0(s)^{p_1/p_0}}\,ds\right)^{1/p_1} \lesssim \frac{W_1(a_t)^{1/p_1}}{W_0(a_t)^{1/p_0}},$$

and, hence, we have proved (a).

(b) We need to check (10) and (11). Now,

$$\left(\int_0^{a_t} \frac{w_0(s)}{W_1(s)^{p_0/p_1}}\,ds\right)^{1/p_0} = \left(\int_0^{a_t} \left(\frac{W_0(s)^{\alpha}}{W_1(s)^{\beta}}\right)^{p_0/\beta p_1} \frac{w_0(s)}{W_0(s)^{\alpha p_0/\beta p_1}}\,ds\right)^{1/p_0}$$

$$\lesssim \frac{W_0(a_t)^{1/p_0}}{W_1(a_t)^{1/p_1}} = t,$$

and, similarly,

$$\left(\frac{W_1(a_t)}{W_0(a_t)^{p_1/p_0}} + \int_{a_t}^{\infty} \frac{w_1(s)}{W_0(s)^{p_1/p_0}}\,ds\right)^{1/p_1} \lesssim \frac{W_1(a_t)^{1/p_1}}{W_0(a_t)^{1/p_0}} = \frac{1}{t},$$

and (b) follows. ▮

Remark 1. If the spaces $\Lambda^{p_i}(w_i)$ are Banach spaces (see [28]) and if the underlaying measure space is resonant then it holds that (see [6, p. 67]),

$$w_i(t) \lesssim \frac{W_i(t)}{t}, \qquad \text{a.e.}$$

Moreover, condition (13) is a kind of separation property of the spaces that can be equivalently described in terms of indices (see [26]). Rearrangement invariant spaces with this kind of separation are considered in [19], [22] and [26].

We end this section by discussing the possibility to express the *K*-functional for a pair of weighted Lorentz spaces in terms of the distribution function instead of the decreasing rearrangement function.

Define $Sf := \lambda_f^\mu$, where $\lambda_f^\mu(s) := \mu\{x\colon |f(x)| > s\}$. The weighted Lorentz spaces $\Lambda^p(w)$, see [10], can also be seen as the set of all measurable functions on a measure space Ω, for which

$$\|f\|_{\Lambda^p(w)} := \left(\int_0^\infty y^{p-1} W(\lambda_f(y))\, dy\right)^{1/p} < \infty.$$

Hence, to study the pair $(\Lambda^{p_0}(w_0), \Lambda^{p_1}(w_1))$, we may also consider $\bar{E} := (E_0, E_1)$, where

$$\|f\|_{E_i} := \left(\int_0^\infty y^{p_i-1} W_i(f(y))\, dy\right)^{1/p_i}.$$

In order to have no problem with the S-divisibility we assume that

$$\int_0^\infty w_i(s)\, ds = +\infty,$$

and then $\Lambda^{p_i}(w_i) = (L_0^*)_{E_i;\, S}$. To prove the S-divisibility property, let $f \in L_0^*$ and $\varphi_i \in D$ be such that $\lambda_f^\mu \leqslant \varphi_0 + \varphi_1$. Then

$$f_\mu^* \leqslant \lambda_{\varphi_0}^m(\cdot/2) + \lambda_{\varphi_1}^m(\cdot/2),$$

where m is the Lebesgue measure. By the $f \mapsto f^*$ divisibility of L_0^* with respect to D and all pairs of lattices $\bar{E}$, we have that there exist f_i such that $f = f_0 + f_1$ and $(f_i)_\mu^* \leqslant \lambda_{\varphi_i}^m(\cdot/2) = \lambda_{2\varphi_i}^m$ and hence

$$Sf_i = \lambda_{f_i}^\mu \leqslant (2\varphi_i)^* \leqslant 2\varphi_i,$$

and the S-divisibility property is proved.

Therefore, by Theorem 2, we obtain that

$$K(t, f; \Lambda^{p_0}(w_0), \Lambda^{p_1}(w_1)) \approx K(t, \lambda_f^\mu; E_0^D, E_1^D).$$

Similar conditions to (10) and (11) can also be obtained in this setting. However, they are not so easy to handle in general.

3.4. *A Final Application of Theorem* 3

By using Theorem 3 we obtain equivalence formulas for the K-functional for elements f from a cone P of positive functions, for a certain class of $(n+1)$-tuples of lattices. The assumption (4) is clearly satisfied, e.g., for the cone P of all positive functions. Hence to obtain a formula for the K-functional in this case, it suffices to find a family of sets $\{B(t)\}$ satisfying certain conditions.

Here we show how this technique can be used to obtain a description of the K-functional for weighted l_p-spaces if there are some separation between the weights. (See also [18].)

We define $l_r(w)$ as the set of all sequences $x=(x_n)=(..., x_{-1}, x_0, x_1, ...)$ for which

$$\|x\|_{l_r(w)} := \left(\sum_{i=-\infty}^{\infty} [x_i w_i]^r \right)^{1/r} < \infty,$$

where $0 < r \leqslant \infty$, $w=(w_i)$ and $w_i > 0$.

THEOREM 13. *Let* $0 < p, q \leqslant \infty$, $u=(u_i)$, $u_i>0$, *and* $v=(v_i)$, $v_i>0$. *If*

$$\sup_k \operatorname{card}\{i: 2^{k-1} \leqslant u_i/v_i < 2^k\} < \infty, \tag{14}$$

then

$$K(t, x; l_p(u), l_q(v)) \approx \| |x| \chi_A \|_{l_p(u)} + t \, \| |x| \chi_{A^c} \|_{l_q(v)},$$

where $A = \{i \in \mathbb{Z}: u_i/v_i < t\}$.

Proof. Since $l_p(u)$ and $l_q(v)$ are lattices it holds that

$$\begin{aligned} K(t, x; l_p(u), l_q(v)) &= K(t, |x|; l_p(u), l_q(v)) \\ &= \inf\{ \|y\|_{l_p(u)} + t \, \|z\|_{l_q(v)}: |x| \leqslant y+z, \ y, z \geqslant 0\}, \end{aligned}$$

see [8], and hence, we may use Theorem 3 with P as the cone of all positive sequences. In this case, the function $h^P(m, n)$ clearly is the Kronecker delta function $\delta_{m,n}$ which is equal to one if $m=n$ and zero otherwise. Now, by definition, $(g_0)_m := \|h^P(m, \cdot)\|_{l_p(u)}$ and $(g_1)_m := \|h^P(m, \cdot)\|_{l_q(v)}$, which gives that $g_0 = u$ and $g_1 = v$, respectively. It remains to check that

$$\|\chi_A / v\|_{l_p(u)} \lesssim t \qquad \text{and} \qquad \|\chi_{A^c}/u\|_{l_q(v)} \lesssim 1/t.$$

Let A_k denote the set $\{i: 2^{k-1} \leqslant u_i/v_i < 2^k\}$ and $C := \sup_k \operatorname{card} A_k$, which is finite by assumption. Choose k_0 such that $2^{k_0-1} \leqslant t < 2^{k_0}$ and it follows that $A \subset \bigcup_{k=-\infty}^{k_0} A_k$ and $A^c \subset \bigcup_{k=k_0-1}^{\infty} A_k$. Hence,

$$\begin{aligned} \|\chi_A / v\|_{l_p(u)} &= \left(\sum_{i \in A} [u_i/v_i]^p \right)^{1/p} \leqslant \left(\sum_{k=-\infty}^{k_0} \sum_{i \in A_k} [u_i/v_i]^p \right)^{1/p} \\ &\leqslant C \left(\sum_{k=-\infty}^{k_0} 2^{kp} \right)^{1/p} \lesssim 2^{k_0 - 1} \leqslant t, \end{aligned}$$

and

$$\|\chi_{A^c}/u\|_{l_q(v)} = \left(\sum_{i \in A^c} [v_i/u_i]^q \right)^{1/q} \leqslant \left(\sum_{k=k_0-1}^{\infty} \sum_{i \in A_k} [v_i/u_i]^q \right)^{1/q}$$

$$\leqslant C \left(\sum_{k=k_0-1}^{\infty} 2^{(1-k)q} \right)^{1/q} \lesssim 2^{-k_0} < 1/t,$$

and the proof is complete. ∎

Note the fact that the "almost optimal" decomposition is independent of p and q. A special case of weights satisfying (14) is if there is an $a \in (0, 1)$ such that $i \mapsto a^i u_i/v_i$ is increasing.

The computation of $(l_p(u), l_q(v))_{\vartheta, r}$, for this type of weights in the case $p, q, r \geqslant 1$ can be found in [14]. However, the proof does not use the K-functional.

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